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Some Best Constants in the Landau Inequality on a Finite Interval

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An additive form of the Landau inequality for $f \in W^n_\infty[-1, 1]$,

$$\|f^{(m)}\| \leqslant \frac{1}{c^m}\left(1-\frac{m}{n}\right) T_n^{(m)}(1)\, \|f\| + \frac{c^{n-m}}{2^{n-1}n!}\,\frac{m}{n}\, T_n^{(m)}(1)\, \|f^{(n)}\|,$$

is proved for $0 < c \leqslant (\cos(\pi/2n))^{-2}$, $1 \leqslant m \leqslant n-1$, with equality for $f(x) = T_n(1+(x-1)/c)$, $1 \leqslant c \leqslant (\cos(\pi/2n))^{-2}$, where T_n is the Chebyshev polynomial. From this follows a sharp multiplicative inequality,

$$\|f^{(m)}\| \leqslant (2^{n-1}n!)^{-m/n}\, T_n^{(m)}(1)\, \|f\|^{1-m/n}\, \|f^{(n)}\|^{m/n}$$

for $\|f^{(n)}\| \geqslant \sigma \|f\|$, $2^{n-1}n! \cos^{2n}(\pi/2n) \leqslant \sigma \leqslant 2^{n-1}n!$, $1 \leqslant m \leqslant n-1$. For these values of σ, the result confirms Karlin's conjecture on the Landau inequality for intermediate derivatives on a finite interval. For the proof of the additive inequality a Duffin and Schaeffer-type inequality for polynomials is shown.

1. INTRODUCTION AND STATEMENT OF RESULTS

The Landau or Landau–Kolmogorov inequality,

$$\|f^{(m)}\|_\infty \leqslant c_{n,m}\, \|f\|_\infty^{1-m/n}\, \|f^{(n)}\|_\infty^{m/n} \tag{1.1}$$

for functions $f \in W^n_\infty(I)$, $I = \mathbf{R}$, $\mathbf{R}_+$ or $[-1, 1]$, where the problem is to find the best possible constants $C^\infty_{n,m}$, $C^+_{n,m}$, and $C_{n,m}$ respectively, was introduced in 1913 by E. Landau [LAN], who found $C^\infty_{2,1} = \sqrt{2}$ and $C^+_{2,1} = 2$.

The problem for $I = \mathbf{R}$ was solved in 1938 by A. N. Kolmogorov [KOL], who also found extremal functions, so-called Euler splines.

The problem for $I = \mathbf{R}_+$ was studied in 1914 by J. Hadamard [HAD] and in 1955 by A. P. Matorin [MAT], who found that $C^+_{n,m} \leqslant (2^{n-1}n!)^{-m/n}\, T_n^{(m)}(1) = M_{n,m}$, with equality for $n = 2$ and $n = 3$. In 1970

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I. J. Schoenberg and A. Cavaretta [S&C], solved the problem for $\mathbf{R}_+$ and showed that $C^+_{n,m} < M_{n,m}$ when $n \geqslant 4$. More generally than in (1.1), we can consider as a Landau inequality, any sharp inequality of the form

$$\|f^{(m)}\| \leqslant F(\|f\|, \|f^{(n)}\|)$$

or

$$\|f^{(m)}\| \leqslant F_\sigma(\|f\|, \|f^{(n)}\|), \qquad \text{with} \quad \|f^{(n)}\| \geqslant \sigma\,\|f\| \quad \text{or} \quad \|f^{(n)}\| \leqslant \sigma\,\|f\|,$$

where $F(x, y)$ is homogeneous in x and y, i.e., where $xF'_x + yF'_y = F(x, y)$. Here and in the following we write $\|f\|_\infty = \|f\|$. We will study the cases $F(x, y) = Ax + By$ in Theorem 1 and $F(x, y) = Cx^{1-m/n}y^{m/n}$ in Theorem 2.

When $I = [-1, 1]$ the situation is somewhat different in (1.1) compared to $I = \mathbf{R}$ or $\mathbf{R}_+$. Then there is no common constant for all $W^n_\infty[-1, 1]$, but only for each $\sigma > 0$, a best constant $c_{n,m} = C_{n,m}(\sigma)$, such that (1.1) holds for every $f \in W^n_\infty[-1, 1]$ with $\|f^{(n)}\| \geqslant \sigma \|f\|$. See the book of R. A. DeVore and G. G. Lorentz [D&L, pp. 38–39]. In Theorem 2 we show that $C_{n,m}(\sigma) = M_{n,m}$, $1 \leqslant m \leqslant n-1$, when $2^{n-1} n! \cos^{2n}(\pi/2n) \leqslant \sigma \leqslant 2^{n-1} n!$. In obtaining this result we lean on the results of A. Yu. Shadrin in [SH1, SH3].

The result in Theorem 2 was proved for $n = 2$ in 1975 by C. K. Chui and P. W. Smith [C&S] and for $n = 3$ in 1982 by M. Sato [SAT]. In 1993 the result was announced for $m = n-1$ and $m = n-2$ by A. Yu. Shadrin in [SH2], where also more results and references are given. Up to then best known, and also some best possible, constants in connection with this problem were given in 1976 and 1990 by H. Kallioniemi in [KA1, KA2].

THEOREM 1. *Let $f \in W^n_\infty[-1, 1]$. Then, for every c, $0 < c \leqslant (\cos(\pi/2n))^{-2}$, and for every m, $1 \leqslant m \leqslant n-1$,*

$$\|f^{(m)}\| \leqslant \frac{1}{c^m}\left(1 - \frac{m}{n}\right) T_n^{(m)}(1)\,\|f\| + \frac{c^{n-m}}{2^{n-1} n!}\,\frac{m}{n}\,T_n^{(m)}(1)\,\|f^{(n)}\|. \tag{1.2}$$

THEOREM 2. *Let $f \in W^n_\infty[-1, 1]$, where $n \geqslant 2$, and suppose that*

$$\|f^{(n)}\| \geqslant \sigma\,\|f\|, \qquad 2^{n-1} n! \cos^{2n}(\pi/2n) \leqslant \sigma \leqslant 2^{n-1} n!. \tag{1.3}$$

Then for every m, $1 \leqslant m \leqslant n-1$, $C_{n,m}(\sigma) = M_{n,m}$, i.e.,

$$\|f^{(m)}\| \leqslant (2^{n-1} n!)^{-m/n}\, T_n^{(m)}(1)\,\|f\|^{1-m/n}\,\|f^{(n)}\|^{m/n}, \tag{1.4}$$

with equality for $f(x) = T_n(1 + (x-1)/c)$, $1 \leqslant c \leqslant (\cos(\pi/2n))^{-2}$.

Remark 1.1. The minimum of the right hand side of (1.2) (f fixed, $c>0$) occurs when $c^n = 2^{n-1} n!\, \|f\| / \|f^{(n)}\|$, and this value of c turns (1.2) into (1.4). Hence Theorem 2 follows at once from Theorem 1, for those functions f for which the inequalities (1.2) and (1.3) hold.

The equality for the mentioned functions is obvious.

A second setting of the Landau problem is that of determining

$$\phi_m(\sigma, \xi) = \sup_f \{ |f^{(m)}(\xi)| : \|f\| = 1,\ \|f^{(n)}\| \leqslant \sigma \}, \qquad -1 \leqslant \xi \leqslant 1,\ \sigma \geqslant 0,$$

and

$$\phi_m(\sigma) = \sup_f \{ \|f^{(m)}\| : \|f\| = 1,\ \|f^{(n)}\| \leqslant \sigma \}, \qquad \sigma \geqslant 0. \tag{1.5}$$

In 1978, A. Pinkus [PIN], showed the existence of perfect splines $P_{\sigma, \xi}(x)$ such that $\|P_{\sigma, \xi}\| = 1$, $\|P^{(n)}_{\sigma, \xi}\| = \sigma$, and such that

$$|P^{(m)}_{\sigma, \xi}(\xi)| = \phi_m(\sigma, \xi),$$

which gives

$$\phi_m(\sigma) \leqslant C_{n, m}(\sigma)\, \sigma^{m/n}.$$

When $\sigma > 0$, $C_{n, m}(\sigma)$ is decreasing and $\phi_m(\sigma)$ is increasing. For $\|f\| = 1$, $\|f^{(n)}\| = \sigma = \alpha^n 2^{n-1} n!$, with $\alpha \geqslant \cos^2(\pi/2n)$, Theorem 2 gives

$$\|f^{(m)}\| \leqslant \alpha^m T_n^{(m)}(1) = T_n^{(m)}(1)(\sigma/2^{n-1} n!)^{m/n}, \tag{1.6}$$

with equality for $f(x) = T_n(1 + (x-1)/c)$, $1 \leqslant c \leqslant 1/\cos^2(\pi/2n)$, i.e., for $\cos^{2n}(\pi/2n)\, 2^{n-1} n! \leqslant \sigma \leqslant 2^{n-1} n!$. When $\|f^{(n)}\| = \sigma = 2^{n-1} n! (\beta \cos^2(\pi/2n))^n$, $0 < \beta \leqslant 1$, Theorem 1 with $c = 1/\cos^2(\pi/2n)$, gives

$$\|f^{(m)}\| \leqslant \cos^{2m}(\pi/2n)\, T_n^{(m)}(1) \left(1 - \frac{m}{n} + \frac{m}{n} \beta^n\right). \tag{1.7}$$

Thus (1.6) and (1.7) yield

$$\phi_m(\sigma) \leqslant T_n^{(m)}(1)(2^{n-1} n!)^{-m/n} \sigma^{m/n}, \qquad \sigma > 2^{n-1} n!, \tag{1.8}$$

$$\phi_m(\sigma) = T_n^{(m)}(1)(2^{n-1} n!)^{-m/n} \sigma^{m/n}, \qquad \cos^{2n}(\pi/2n)\, 2^{n-1} n! \leqslant \sigma \leqslant 2^{n-1} n! \tag{1.9}$$

and

$$T_{n-1}^{(m)}(1) \leqslant \phi_m(\sigma) \leqslant \cos^{2m}(\pi/2n)\, T_n^{(m)}(1)\left(1-\frac{m}{n}+\frac{m}{n}\beta^n\right),$$
$$0<\sigma=2^{n-1}n!(\beta\cos^2(\pi/2n))^n, \qquad 0<\beta\leqslant 1. \tag{1.10}$$

In 1976 S. Karlin [KAR, p. 423] conjectured that, with n fixed, for each $\sigma>0$,

$$\phi_m(\sigma)=Z^{(m)}(1,\sigma),$$

where $Z(x,\sigma)$ is the unique perfect spline of degree n with r nodes, $-1<x_1<\cdots<x_r<1$,

$$Z(x,\sigma)=c\left(x^n+2\sum_{i=1}^{r}(-1)^i(x-x_i)_+^n\right)+\sum_{i=1}^{n-1}a_ix^i,$$

with $n+r+1$ (in this case $Z(x,\sigma)$ is denoted by $T_{n,r}(x)$, $T_{n,0}(x)=T_n(x)$) or $n+r$ points of equioscillation, made unique by the requirements $\|Z(\cdot,\sigma)\|=1$, $\|Z^{(n)}(\cdot,\sigma)\|=\sigma$, $Z(1,\sigma)=1$, and $Z^{(n)}(1,\sigma)=\sigma$. We have $Z(x,\sigma)=T_n(1+(x-1)/c)$ for the values of $\sigma=2^{n-1}n!/c^n$, considered in Theorem 2, which confirms Karlin's conjecture in this case. It is true for all $\sigma>0$ when $n=2$ [C&S], and $n=3$ [SAT].

The inequality (1.4) is studied in [C&S] for $\sigma=\sigma_{n,r}=T_{n,r}^{(n)}(1)$, $n=4$, $0\leqslant r\leqslant 2$, $5\leqslant n\leqslant 6$, $0\leqslant r\leqslant 4$, $1\leqslant m\leqslant n-1$.

This investigation shows that if Karlin's conjecture is true, inequality (1.4) is not sharp, although rather strong, for the mentioned values of σ.

A third setting of the Landau inequality is the problem of finding exact constants in inequalities of the form

$$|f^{(m)}(\xi)|\leqslant A(\xi)\,\|f\|+B(\xi)\,\|f^{(n)}\|$$

and

$$\|f^{(m)}\|\leqslant A\,\|f\|+B\,\|f^{(n)}\|, \qquad f\in W_\infty^n[-1,1],$$

where $A\geqslant T_{n-1}^{(m)}(1)$, and $B=B(A)$ is a convex function. Here Theorem 1 gives an upper bound for $B=B(A)$, $A\geqslant A_m$, and for some A the exact value of $B(A)$.

Before formulating Theorem 3 we define the following sets of polynomials by

$$B_n=\{p\in P_n: |p(x)|\leqslant 1,\ |x|\leqslant 1\}, \tag{1.11}$$

$$C_n=\{p\in P_n: |p(y_k)|\leqslant 1,\ y_k=\cos(k\pi/n),\ 0\leqslant k\leqslant n\}, \tag{1.12}$$

and

$$D_{n-1} = \{ p \in P_{n-1} : |p(y_k)| \leqslant 1,\ y_k = \cos(k\pi/n),\ 0 \leqslant k \leqslant n-1 \}, \tag{1.13}$$

where P_n is the set of all polynomials of degree at most n.

It was proved by V. A. Markov [MAR] that, if $p \in B_n$, then

$$\|p^{(m)}\| \leqslant \|T_n^{(m)}\|, \qquad 1 \leqslant m \leqslant n. \tag{1.14}$$

This result was generalized by Duffin and Schaeffer [D&S], who proved that (1.14) holds also if $p \in C_n$. The following theorem is a result of a similar kind.

THEOREM 3. *If $p \in D_{n-1}$, then for every x, $0 \leqslant x \leqslant 1$, and for every m, $1 \leqslant m \leqslant n-1$,*

$$|p^{(m)}(x)| \leqslant (1 - m/n)\, \|T_n^{(m)}\|. \tag{1.15}$$

We have equality in (1.15) for $x = 1$, $1 \leqslant m \leqslant n-1$, when $p = Q$ is the polynomial of degree $n-1$ interpolating T_n at y_k, $0 \leqslant k \leqslant n-1$, i.e., when $Q(x) = T_n(x) - (x-1)\, T_n'(x)/n$.

2. PREREQUISITES FOR THE PROOFS

The mth derivative of the Chebyshev polynomial

$$T_n(x) = 2^{n-1} x^n - \cdots = T_n(\cos\theta) = \cos n\theta, \qquad -1 \leqslant x \leqslant 1,\ 0 \leqslant \theta \leqslant \pi,$$

satisfies the differential equation

$$(1 - x^2)\, T_n^{(m+2)} - (2m+1)\, x T_n^{(m+1)} + (n^2 - m^2)\, T_n^{(m)} = 0. \tag{2.1}$$

Since $T_n(y_k) = \cos(nk\pi/n) = (-1)^k$, it follows that $y_k = \cos(k\pi/n)$, $0 \leqslant k \leqslant n$, are the extreme points of T_n, and that

$$T_n'(x) = 2^{n-1} n (x - y_1) \cdots (x - y_{n-1}). \tag{2.2}$$

From (2.1) we obtain the recursion formula

$$T_n^{(m+1)}(1) = \frac{n^2 - m^2}{2m+1}\, T_n^{(m)}(1). \tag{2.3}$$

Next we list four theorems and one lemma, which will be useful in the sequel.

In his paper [SH3], A. Yu. Shadrin proved the following theorem on derivative error bounds for Lagrange interpolation.

THEOREM A. *Let $p_{n-1}(x)$ be the polynomial of degree at most $n-1$, that interpolates $f \in W^n_\infty[a, b]$ at the points $t_0, ..., t_{n-1}$: $a \leqslant t_{n-1} < t_{n-2} < \cdots < t_1 < t_0 \leqslant b$ and set $\omega(x) = \prod_{k=0}^{n-1} (x - t_k)$.*

If $\omega_j(x) = \omega(x)/(x - t_j)$, $0 \leqslant j \leqslant n-1$, then

$$p_{n-1}(x) = \sum_{j=0}^{n-1} \frac{f(t_j)}{\omega'(t_j)} \omega_j(x).$$

Let

$$\omega_0^{(m)}(x) = c \prod_{j=1}^{n-1-m} (x - \alpha_j), \qquad \alpha_{n-1-m} < \cdots < \alpha_2 < \alpha_1,$$

and let

$$\omega_{n-1}^{(m)}(x) = c \prod_{j=1}^{n-1-m} (x - \beta_j), \qquad \beta_{n-1-m} < \cdots < \beta_2 < \beta_1, 1 \leqslant m \leqslant n-2.$$

If $\alpha_0 = t_0$, $\beta_0 = b$, $\alpha_{n-m} = a$, $\beta_{n-m} = t_{n-1}$, $I_{n,m} = \bigcup_{j=0}^{n-m} [\alpha_j, \beta_j]$, and $J_{n,m} = \bigcup_{j=1}^{n-m} (\beta_j, \alpha_{j-1})$, then

$$\sup_{\|f^{(n)}\| \leqslant 1} |f^{(m)}(x) - p_{n-1}^{(m)}(x)| = \frac{1}{n!} |\omega^{(m)}(x)|, \qquad x \in I_{n,m}, \tag{2.4}$$

and

$$\sup_{\|f^{(n)}\| \leqslant 1} |f^{(m)}(x) - p_{n-1}^{(m)}(x)| \leqslant \frac{1}{n!} \max\{|\omega^{(m)}(\beta_j)|, |\omega^{(m)}(\alpha_{j-1})|\}, \quad x \in (\beta_j, \alpha_{j-1}) \subset J_{n,m}. \tag{2.5}$$

From Theorem A we obtain the estimate

$$\sup_{\beta \leqslant x \leqslant b} |f^{(m)}(x) - p_{n-1}^{(m)}(x)| \leqslant \frac{\|f^{(n)}\|}{n!} \sup_{\beta \leqslant x \leqslant b} |\omega^{(m)}(x)| \tag{2.6}$$

for any $f \in W^n_\infty[a, b]$, when β is a point in $I_{n,m}$.

In [SH1], A. Yu. Shadrin proved (1.14) for polynomials in C_n using the following theorem.

THEOREM B. *Let $q \in P_n$ have n distinct zeros in the interval $[-1, 1]$. Let u_j, $1 \leqslant j \leqslant n-1$, be the zeros of q' and set $u_0 = 1$ and $u_n = -1$. If a polynomial $P \in P_n$ satisfies the inequality*

$$|P(u_j)| \leqslant |q(u_j)|, \qquad 0 \leqslant j \leqslant n,$$

then for each m, $1 \leqslant m \leqslant n$, and for each x, $-1 \leqslant x \leqslant 1$,

$$|P^{(m)}(x)| \leqslant \max \left\{ |q^{(m)}(x)|, \left| \frac{1}{m}(x^2-1)\, q^{(m+1)}(x) + x q^{(m)}(x) \right| \right\}.$$

If $q(x) = T_n(x)$, $u_j = y_j$, and if $|P(y_j)| \leqslant 1$, $0 \leqslant j \leqslant n$, the last inequality becomes

$$|P^{(m)}(x)| \leqslant \max \left\{ |T_n^{(m)}(x)|, \left| \frac{1}{m}(x^2-1)\, T_n^{(m+1)}(x) + x T_n^{(m)}(x) \right| \right\}. \tag{2.7}$$

THEOREM C (Sonin–Pólya). *If p, p', q, and q' are continuous in an interval J, $p, q > 0$ and if $(p(x)\, q(x))' \leqslant 0$, for $x \in J$, then $|y(x_i)|$ is increasing on the set of local extreme points x_i for y, when y is a non-trivial solution of the differential equation*

$$(p(x)\, y')' + q(x)\, y = 0,$$

which is of self-adjoint form.

The Sonin–Pólya theorem is proved by studying the function

$$F(x) = y^2 + \frac{p(x)}{q(x)}(y')^2 = y^2 + \frac{1}{p(x)\, q(x)}(p(x)\, y')^2, \tag{2.8}$$

which, using the differential equation, gives

$$F'(x) = -\frac{(p(x)\, q(x))'}{(p(x)\, q(x))^2}(p(x)\, y')^2 \geqslant 0.$$

Thus F is increasing in J, and the theorem follows.

THEOREM D. (The Sturm Comparison Theorem). *If $q(x)$ and $r(x)$ are continuous on $[a, b]$, $q(x) \geqslant r(x)$, $q \neq r$, if $y(x)$ and $z(x)$ are non-trivial solutions of the differential equations*

$$y'' + q(x)\, y = 0$$

and

$$z'' + r(x)\, z = 0,$$

and if $z(a) = z(b) = 0$, *then there exists at least one point* x_0, $a < x_0 < b$, *such that* $y(x_0) = 0$.

For a proof of Theorem D, see [SIM].

If we write the differential equation (2.1) in the form

$$T_n^{(m+2)} + p(x)\, T_n^{(m+1)} + q(x)\, T_n^{(m)} = 0,$$

it can be transformed to the form

$$u'' + S_m(x)\, u = 0,$$

where $T_n^{(m)} = u\varphi$, $\varphi(x) = e^{-\int (p(x)/2)\, dx} > 0$, and $S_m(x) = q(x) - \frac{1}{4}(p(x))^2 - \frac{1}{2}p'(x)$. We see that $T_n^{(m)}$ and u have the same zeros. The differential Eq. (2.1) gives

$$\begin{aligned} S_m(x) &= \frac{n^2 - m^2}{1 - x^2} - \frac{1}{4}(2m+1)^2 \frac{x^2}{(1-x^2)^2} + \frac{1}{2}(2m+1) \frac{1 + x^2}{(1-x^2)^2} \\ &= \frac{n^2(1-x^2) - (m-0.5)^2 + 0.75 + 0.25x^2}{(1-x^2)^2} \end{aligned} \tag{2.9}$$

LEMMA A. (a) *If* T_n *is the Chebyshev polynomial of degree n, then*

$$|T_n^{(m)}(x)| \leqslant T_n^{(m)}(1), \qquad |x| \leqslant 1,\ 1 \leqslant m \leqslant n, \tag{2.10}$$

and

$$|T_n'(x)| \leqslant n/\sqrt{1-x^2}, \qquad |x| < 1. \tag{2.11}$$

(b) (*Markov*) *If* $p(x) = (x - a_1) \cdots (x - a_n)$, $q(x) = (x - b_1) \cdots (x - b_n)$, $p \neq q$, *where* $a_1 < a_2 < \cdots < a_n$, $b_1 < b_2 < \cdots < b_n$, $a_1 \leqslant b_1 \leqslant a_2 \leqslant b_2 \leqslant \cdots \leqslant a_n \leqslant b_n$, *then, if* $s_1 < \cdots < s_{n-1}$ *are the zeros of* p' *and* $t_1 < \cdots < t_{n-1}$ *are the zeros of* q', *we have*

$$s_1 < t_1 < s_2 < t_2 < \cdots < s_{n-1} < t_{n-1}.$$

It then follows that the zeros of $p^{(m)}$ *and* $q^{(m)}$, $2 \leqslant m \leqslant n-1$, *interlace in the same way as those of* p' *and* q'.

For a proof of Lemma A see the book of T. J. Rivlin [RIV].

Remark 2.1. Lemma A(b) is true also if $p(x)=(x-a_2)(x-a_3)\cdots(x-a_n)$, and if $b_1 \leqslant a_2 \leqslant b_2 \leqslant \cdots \leqslant a_n \leqslant b_n$. This is proved in a similar way, interpolating $p(x)$ instead of $p(x)-q(x)$ at the points b_j, $1 \leqslant j \leqslant n$.

We denote the relation between the polynomials p and q in Lemma A(b) or in Remark 2.1 by $p \prec q$. The relation $\prec$ is not transitive, but if $p_1 \prec p_2 \prec \cdots \prec p_k$ and if moreover $p_1 \prec p_k$, then $p_i \prec p_j$ for $1 \leqslant i < j \leqslant k$.

The main tools in our investigation will be $T_n(x)$ and the following polynomials

$$L_n(x)=(x-1)\,T'_n(x)=2^{n-1}n(x-1)(x-y_1)\cdots(x-y_{n-1}), \tag{2.12}$$

$$L_{n,j}(x)=\frac{L_n(x)}{(x-y_j)}, \qquad 1 \leqslant j \leqslant n-1, \tag{2.13}$$

and

$$Q(x)=T_n(x)-\frac{1}{n}(x-1)\,T'_n(x)=T_n(x)-\frac{1}{n}L_n(x). \tag{2.14}$$

With $\omega(x)=(x-1)\,T'_n(x)=L_n(x)$, we have for $1 \leqslant j \leqslant n-1$,

$$\begin{aligned}
\omega_0(x)&=\omega(x)/(x-1)=T'_n(x)=L_{n,0}(x),\\
\omega_j(x)&=\omega(x)/(x-y_j)=L_{n,j}(x),\\
\omega'(1)&=T'_n(1)=n^2,
\end{aligned}$$

and

$$\omega'(y_j)=(y_j-1)\,T''_n(y_j)=(y_j^2-1)\,T''_n(y_j)/(1+y_j)=n^2(-1)^j/(1+y_j).$$

Then

$$\begin{aligned}
p_{n-1}(x)&=\sum_{j=0}^{n-1}\frac{f(y_j)\,\omega(x)}{\omega'(y_j)(x-y_j)}\\
&=\frac{f(1)}{n^2}\,T'_n(x)+\sum_{j=1}^{n-1} f(y_j)\,\frac{1+y_j}{n^2}\,(-1)^j\,L_{n,j}(x)
\end{aligned} \tag{2.15}$$

is the Lagrange interpolation polynomial of degree $\leqslant n-1$, interpolating f at the points 1, $y_1, ..., y_{n-1}$. This interpolation will be used in the proof of Theorem 3, where $f \in D_{n-1}$ and $p_{n-1}=f$, and in the proof of Theorem 1.

Denote the largest zeros of $T_n^{(m)}(x)$, $L_n^{(m)}(x)$, $L_{n,j}^{(m)}(x)$, and $Q^{(m)}(x)$ by ω_m, λ_m, $\mu_{j,m}$, and q_m, respectively, and denote the smallest positive zero of $T_n^{(m)}(x)$ by ω_m^+.

Below we obtain in Lemmas 2–4 estimates for $T_n^{(m)}(x)$, $L_n^{(m)}(x)$, and $Q^{(m)}(x)$, respectively. Lemma 1 gives some control of the zeros of the polynomials mentioned.

LEMMA 1. (a) *Since*

$$L_n' \prec T_n \prec L_n, \tag{2.16}$$

$$T_n' \prec L_{n,1} \prec L_{n,2} \prec \cdots \prec L_{n,n-1} \prec L_n \quad \textit{and} \quad T_n' \prec L_n, \tag{2.17}$$

$$T_n' \prec L_n' \prec L_{n,n-1} \quad \textit{and} \quad T_n' \prec L_{n,n-1}, \tag{2.18}$$

$$T_n' \prec Q \prec T_n, \tag{2.19}$$

we have for $1 \leqslant m \leqslant n-2$,

$$\omega_{m+1} < \lambda_{m+1} < \omega_m < \lambda_m, \tag{2.20}$$

$$\omega_{m+1} < \mu_{1,m} < \mu_{2,m} < \cdots < \mu_{n-1,m} < \lambda_m, \tag{2.21}$$

$$\omega_{m+1} < \lambda_{m+1} < \mu_{n-1,m}, \tag{2.22}$$

$$\omega_{m+1} < q_m < \omega_m. \tag{2.23}$$

(b) *If* $n-m \geqslant 2$ *is even, then*

$$0 < \omega_m^+ \leqslant 1/\sqrt{2n-2}. \tag{2.24}$$

Proof. (a) Combining (2.12) and (2.1) for $m=0$, we obtain

$$L_n'(x) = T_n'(x) + (x-1)\,T_n''(x) = (T_n'(x) + n^2 T_n(x))/(x+1),$$

which gives $\operatorname{sgn}(L_n'(x_k)) = (-1)^{k-1}$, where x_k, $1 \leqslant k \leqslant n$, are the zeros of $T_n(x)$, and the relation $L_n' \prec T_n$ follows. The other relations in (2.16)–(2.19) are obvious. Hence the results follow from Lemma A(b).

(b) The Sturm comparison theorem applied to (2.9), where $S_m(x) > S_{m+2}(x)$, implies that $T_n^{(m)}$ has at least one, i.e., at least one positive zero in $[-\omega_{m+2}^+, \omega_{m+2}^+]$, since $T_n^{(m)}$ is even. This yields $\omega_m^+ < \omega_{m+2}^+ < \cdots < \omega_{n-2}^+ = 1/\sqrt{2n-2}$, where the equality and hence (2.24) follows from

$$T_n^{(n-2)}(x) = c(x^2 - 1/(2n-2)).$$

LEMMA 2. *If* $1 \leqslant m \leqslant n-2$, *then*

$$|T_n^{(m)}(x)| \leqslant \frac{T_n^{(m)}(1)}{2m+1}, \qquad |x| \leqslant \omega_m. \tag{2.25}$$

Proof. If $y = T_n^{(m)}$, $1 \leqslant m \leqslant n-2$, then y satisfies the differential Eq. (2.1), or in self-adjoint form,

$$(y'(1-x^2)^{m+1/2})' + (n^2-m^2)(1-x^2)^{m-1/2}\, y = 0.$$

The expression $p(x)\, q(x) = (n^2-m^2)(1-x^2)^{2m}$, in Theorem C, is decreasing in the interval $0 \leqslant x \leqslant 1$. Hence

$$F(x) = (T_n^{(m)}(x))^2 + (1-x^2)(T_n^{(m+1)}(x))^2/(n^2-m^2)$$

is increasing, according to the proof of Theorem C, and

$$(T_n^{(m)}(x))^2 \leqslant F(x) \leqslant F(\omega_{m+1}) = (T_n^{(m)}(\omega_{m+1}))^2 \tag{2.26}$$

or

$$|T_n^{(m)}(x)| \leqslant |T_n^{(m)}(\omega_{m+1})|, \qquad 0 \leqslant x \leqslant \omega_{m+1}. \tag{2.27}$$

For $\omega = \omega_{m+1}$, the differential Eq. (2.1), since $T_n^{(m+1)}(\omega) = 0$, gives

$$(1-\omega^2)\, T_n^{(m+2)}(\omega) + (n^2-m^2)\, T_n^{(m)}(\omega) = 0 \tag{2.28}$$

and

$$(1-\omega^2)\, T_n^{(m+3)}(\omega) - (2m+3)\, \omega T_n^{(m+2)}(\omega) = 0. \tag{2.29}$$

From (2.28) and (2.29) we obtain

$$T_n^{(m+2)}(\omega) = \frac{n^2-m^2}{1-\omega^2}\,(-T_n^{(m)}(\omega)) \tag{2.30}$$

and

$$T_n^{(m+3)}(\omega) = \frac{n^2-m^2}{1-\omega^2}\,\frac{(2m+3)\,\omega}{1-\omega^2}\,(-T_n^{(m)}(\omega)). \tag{2.31}$$

Since $T_n^{m+r}(\omega) \geqslant 0$, $r \geqslant 2$, the Taylor formula, together with (2.30) and (2.31) gives

$$\begin{aligned} T_n^{(m+1)}(1) &\geqslant (1-\omega)\, T_n^{(m+2)}(\omega) + \frac{(1-\omega)^2}{2}\, T_n^{(m+3)}(\omega) \\ &= (1-\omega)\,\frac{n^2-m^2}{1-\omega^2}\,(-T_n^{(m)}(\omega)) \\ &\quad + \frac{(1-\omega)^2}{2}\,\frac{n^2-m^2}{1-\omega^2}\,\frac{(2m+3)\,\omega}{1-\omega^2}\,(-T_n^{(m)}(\omega)). \end{aligned} \tag{2.32}$$

Using (2.3) in (2.32), we obtain

$$T_n^{(m+1)}(1) = \frac{n^2 - m^2}{2m+1} T_n^{(m)}(1)$$
$$\geqslant (n^2 - m^2)\left(\frac{1-\omega}{1-\omega^2} + \frac{(2m+3)\,\omega(1-\omega)^2}{2(1-\omega^2)^2}\right)(-T_n^{(m)}(\omega)),$$

or

$$|T_n^{(m)}(\omega)| \leqslant \frac{1}{2m+1}\frac{2(1+\omega)^2}{2+(2m+5)\,\omega} T_n^{(m)}(1), \tag{2.33}$$

which together with (2.27) gives

$$|T_n^{(m)}(x)| \leqslant T_n^{(m)}(\omega) \leqslant \frac{1}{2m+1}\frac{2(1+\omega)^2}{2+(2m+5)\,\omega} T_n^{(m)}(1), \qquad |x| \leqslant \omega_{m+1}. \tag{2.34}$$

Since

$$\frac{2(1+\omega)^2}{2+(2m+5)\,\omega} \leqslant 1$$

when $0 \leqslant \omega \leqslant 1$ and $1 \leqslant m \leqslant n-2$, (2.25) holds for $|x| \leqslant \omega = \omega_{m+1}$.

In the interval $[\omega_{m+1}, \omega_m]$, $|T_n^{(m)}(x)|$ is decreasing, and thus the proof is complete.

Remark 2.1. If $n \geqslant 10$, then $\omega = \omega_2 > y_2 = \cos(2\pi/n) > 3/4$ and $2(1+\omega)^2/(2+7\omega) \leqslant 8/9$, which gives a somewhat better estimate of $|T_n'(x)|$ in (2.34).

LEMMA 3. *Let* $L_n(x) = (x-1)\, T_n'(x)$.

(a) *Then*

$$L_n^{(m)}(1) = m T_n^{(m)}(1), \qquad 1 \leqslant m \leqslant n. \tag{2.35}$$

(b) *If* $1 \leqslant m \leqslant n-2$ *then*

$$|L_n^{(m)}(x)| \leqslant \left(\frac{m}{2m+1} + 1\right) T_n^{(m)}(1), \qquad 0 \leqslant x \leqslant \omega_m, \tag{2.36}$$

and

$$|L_n^{(m)}(x)| \leqslant L_n^{(m)}(1), \qquad 0 \leqslant x \leqslant 1. \tag{2.37}$$

(c) *If* $n-m$ *is odd*, $2 \leqslant m \leqslant n-3$, *then*

$$|L_n^{(m)}(x)| \leqslant L_n^{(m)}(1), \qquad -\omega_{m+1}^{+} \leqslant x \leqslant 1. \tag{2.38}$$

(d) *If* $I_{n,1}$ *is defined as in Theorem A, with* $\omega(x) = L_n(x)/(n2^{n-1})$, *then* $0 \in I_{n,1}$.

Proof. (a) Using the differential Eq. (2.1), we can write

$$\begin{aligned} L_n^{(m)}(x) &= (x-1)\,T_n^{(m+1)}(x) + mT_n^{(m)}(x) \\ &= ((x^2-1)\,T_n^{(m+1)}(x) + m(x+1)\,T_n^{(m)}(x))/(x+1) \\ &= (-(2m-1)\,xT_n^{(m)}(x) + (n^2-(m-1)^2)\,T_n^{(m-1)}(x) \\ &\quad + m(x+1)\,T_n^{(m)}(x))/(x+1) \\ &= ((m(1-x)+x)\,T_n^{(m)}(x) + (n^2-(m-1)^2)\,T_n^{(m-1)}(x))/(x+1). \end{aligned} \tag{2.39}$$

For $x=1$ in the first equality, we obtain (2.35).

(b) From (2.39), using (2.3) and (2.25) we have for $|x| \leqslant \omega_m$, $1 \leqslant m \leqslant n-2$,

$$\begin{aligned} |L_n^{(m)}(x)| &\leqslant \frac{m(1-x)+x}{x+1}\,|T_n^{(m)}(x)| + \frac{n^2-(m-1)^2}{x+1}\,|T_n^{(m-1)}(x)| \\ &\leqslant \frac{m(1-x)+x}{x+1}\,\frac{T_n^{(m)}(1)}{2m+1} + \frac{1}{x+1}\,\frac{n^2-(m-1)^2}{2m-1}\,T_n^{(m-1)}(1) \\ &= \left(\frac{m(1-x)+x}{(2m+1)(x+1)} + \frac{1}{x+1}\right) T_n^{(m)}(1). \end{aligned} \tag{2.40}$$

With

$$g_m(x) = \frac{m(1-x)+x}{(2m+1)(x+1)} + \frac{1}{x+1} \tag{2.41}$$

we see that $g_m'(x) < 0$, $-1 < x < 1$, and thus

$$g_m(x) \leqslant g_m(0) = \frac{m}{2m+1} + 1, \qquad 0 \leqslant x \leqslant 1,$$

which together with (2.40) gives (2.36). Since $\lambda_{m+1} < \omega_m$ and $L_n^{(m)}(x)$ is increasing for $x \geqslant \lambda_{m+1}$ also (2.37) follows, when $m \geqslant 2$.

For $m=1$,

$$L_n'(x)=(x-1)\,T_n''(x)+T_n'(x)=\frac{n^2T_n(x)+T_n'(x)}{1+x},$$

and we need to prove that

$$|n^2T_n(x)+T_n'(x)|\leqslant n^2+n^2x, \qquad 0\leqslant x\leqslant 1.$$

For $n=0$, 1, and 2, (2.37) follows easily. For $n\geqslant 3$, we consider the three intervals $I_1=[\cos(\pi/n), 1]$, $I_2=[\sin(\pi/2n), \cos(\pi/2n)]$, and $I_3=[0, \sin(\pi/2n)]$, which cover $[0, 1]$.

(1) For $x\in[\cos(\pi/n), 1]$, $T_n'(x)$ is convex, since $T_n'''(x)>0$, and thus $0\leqslant T_n'(x)\leqslant n^2x$, since the inequality holds at the endpoints $x=\cos(\pi/n)$ and $x=1$.

(2) For $x\in I_2$ we use inequality (2.11) and $|T_n'(x)|\leqslant n^2x$ will follow if we prove

$$\frac{n}{\sqrt{1-x^2}}\leqslant n^2x \qquad \text{or} \qquad \frac{1}{n^2}\leqslant x^2(1-x^2).$$

The function $x^2(1-x^2)$ attains its minimum on I_2 at the endpoints, that is,

$$\begin{aligned}x^2(1-x^2)&\geqslant \sin^2(\pi/2n)\cos^2(\pi/2n)=(1/4)\sin^2(\pi/n)\\&\geqslant\frac{1}{4}\left(\frac{2}{\pi}\frac{\pi}{n}\right)^2=1/n^2, \qquad n\geqslant 3.\end{aligned}$$

(3) If n is even, and $x\in I_3$, then T_n' and T_n are of opposite sign, that is,

$$|n^2T_n(x)+T_n'(x)|\leqslant \max\{|n^2T_n(x)|, |T_n'(x)|\}\leqslant n^2.$$

If n is odd, $x\in I_3$, we have

$$|T_n'(x)|\leqslant|T_n'(0)|=n, \qquad |T_n(x)|\leqslant|T_n'(0)|\,x=nx,$$

and we can close the case $m=1$ with the inequality

$$\begin{aligned}|n^2T_n(x)+T_n'(x)|&\leqslant(n^2-n)\,|T_n(x)|+|T_n'(x)|+n\,|T_n(x)|\\&\leqslant(n^2-n)+n+n^2x=n^2+n^2x.\end{aligned}$$

(c) From (2.41) we see that $g_m(x_m)=m$ for the decreasing sequence

$$x_m = -(2m^2-2m-1)/(2m^2+2m-1),$$

giving $x_2=-3/11$, $x_3=-11/23$, and $x_4=-23/39$.

(i) When n is even and m is odd, we have by Lemma 1(b)

$$\omega_4^+ < \omega_6^+ < \cdots < \omega_{n-2}^+ = 1/\sqrt{2(n-1)} \leqslant 11/23,$$

when $n \geqslant 6$ and hence

$$x_m \leqslant x_3 = -11/23 \leqslant -\omega_{m+1}^+ \qquad \text{for} \quad 3 \leqslant m \leqslant n-3, \qquad m \text{ odd}.$$

Thus, since $g_m(x)$ is decreasing,

$$|L_n^{(m)}(x)| \leqslant g_m(x)\, T_n^{(m)}(1) \leqslant m T_n^{(m)}(1) = L_n^{(m)}(1) \qquad \text{for} \quad x_m \leqslant x \leqslant 0,$$

i.e., for $-\omega_{m+1}^+ \leqslant x \leqslant 0$, n even, $n \geqslant 6$, $3 \leqslant m \leqslant n-3$, m odd.

(ii) When n is odd and m is even,

$$\omega_3^+ < \omega_5^+ < \cdots < \omega_{n-2}^+ = 1/\sqrt{2(n-1)} \leqslant 3/11,$$

when $n \geqslant 9$, and hence

$$x_m \leqslant x_2 = -3/11 \leqslant -\omega_{m+1}^+ \qquad \text{for} \quad 2 \leqslant m \leqslant n-3, \qquad m \text{ even}.$$

Thus

$$|L_n^{(m)}(x)| \leqslant g_m(x)\, T_n^{(m)}(1) \leqslant m T_n^{(m)}(1) = L_n^{(m)}(1),$$

$-\omega_{m+1}^+ \leqslant x \leqslant 0$, $n \geqslant 9$, n odd, $2 \leqslant m \leqslant n-3$, m even.

When $n=7$, $T_7(x) = 64x^7 - 112x^5 + 56x^3 - 7x$,

$$x_2 = -3/11 < -\omega_3^+ = -(1-(0.6)^{1/2})^{1/2}/2 \approx -0.24$$

and

$$x_4 = -23/39 < -\omega_5^+ = -\sqrt{3}/6.$$

When $n=5$, $T_5(x) = 16x^5 - 20x^3 + 5x$, $\omega_3^+ = \sqrt{2}/4$, and

$$|L_5''(x)| = 40\,|40x^3 - 24x^2 - 9x + 3| \leqslant 400 = L_5''(1), \qquad -\sqrt{2}/4 \leqslant x \leqslant 0.$$

This completes the proof of (c), since we already have (2.37).

(d) Here α_j and β_j in Theorem A are the zeros of T_n'' and $L_{n,n-1}'$, respectively. If n is odd, $T_n''(0)=0$ and $0 \in I_{n,1}$, defined in Theorem A. If n is even,

$\alpha_{n/2} < 0 < \alpha_{n/2-1}$, and $y_{n/2} = 0$. We then have $L_n'(y_{n/2}) = L_n'(0) = n^2 T_n(0) = n^2(-1)^{n/2}$, $\cos(\pi/n)\, L_{n,\,n-1}'(0) = L_n'(0)$ and $T_n''(0) = n^2(-1)^{n/2+1}$. Thus $T_n''(0)$ and $L_{n,\,n-1}'(0)$ have opposite signs, and (2.17) implies that $\beta_{n/2} > 0$. Hence

$$0 \in [\alpha_{n/2}, \beta_{n/2}] \subset I_{n,\,1}.$$

Lemma 4. *Let*

$$Q(x) = T_n(x) - (x-1)\, T_n'(x)/n = T_n(x) - L_n(x)/n. \tag{2.42}$$

(a) *Then* $Q \in D_{n-1}$, $Q(y_j) = T_n(y_j) = (-1)^j$, $0 \leqslant j \leqslant n-1$, *and*

$$Q^{(m)}(1) = (1 - m/n)\, T_n^{(m)}(1). \tag{2.43}$$

(b) *If* $1 \leqslant m \leqslant n-1$, *then*

$$|Q^{(m)}(x)| \leqslant Q^{(m)}(1), \qquad 0 \leqslant x \leqslant 1. \tag{2.44}$$

(c) *If* $n \geqslant 10$, *then*

$$|Q'(x)| \leqslant Q'(1)/3, \qquad 0 \leqslant x \leqslant \omega_1. \tag{2.45}$$

Proof. (a) It is obvious that

$$Q(x) = T_n(x) - \frac{1}{n}(x-1)\, T_n'(x)$$

interpolates T_n at the points y_k, $0 \leqslant k \leqslant n-1$, and that the coefficient of x^n is equal to zero.

Inserting $x = 1$ in

$$Q^{(m)}(x) = \left(1 - \frac{m}{n}\right) T_n^{(m)}(x) + \frac{1-x}{n}\, T_n^{(m+1)}(x), \tag{2.46}$$

we obtain (2.43).

(b) For $0 \leqslant x \leqslant \omega_m$ and $1 \leqslant m \leqslant n-2$, using (2.25), (2.1), and (2.3) we obtain from (2.46)

$$|Q^{(m)}(x)| \leqslant \left(1 - \frac{m}{n}\right) |T_n^{(m)}(x)|$$
$$+ \left| \frac{(2m-1)\, x T_n^{(m)}(x) - (n^2 - (m-1)^2)\, T_n^{(m-1)}(x)}{n(1+x)} \right|$$
$$\leqslant Q^{(m)}(1) \frac{1}{2m+1} + \frac{1}{2} \frac{2m-1}{n} \frac{T_n^{(m)}(1)}{2m+1}$$
$$+ \frac{(n^2 - (m-1)^2)}{n} \frac{T_n^{(m-1)}(1)}{2m-1}$$
$$= \left(\frac{1}{2m+1} + \frac{1}{2} \frac{1}{n-m} \frac{2m-1}{2m+1} + \frac{1}{n-m} \right) Q^{(m)}(1).$$

With

$$G(n, m) = \frac{1}{2m+1} + \frac{1}{2} \frac{1}{n-m} \frac{2m-1}{2m+1} + \frac{1}{n-m},$$

we have

$$G(n, 1) = \frac{1}{3} + \frac{1}{2} \frac{1}{3} \frac{1}{n-1} + \frac{1}{n-1} = \frac{1}{3} + \frac{7}{6} \frac{1}{n-1} \leqslant 1,$$

when $n \geqslant 3$.

When $m \geqslant 2$,

$$G(n, m) \leqslant \frac{1}{5} + \frac{1}{2} \frac{1}{n-m} + \frac{1}{n-m} = \frac{1}{5} + \frac{3}{2} \frac{1}{n-m} \leqslant 1$$

for $n - m \geqslant 2$. In both cases (2.44) follows for $0 \leqslant x \leqslant \omega_m$, $n - m \geqslant 2$.

Since $q_m < \omega_m < 1$, according to (2.19), and since $Q^{(m)}(x)$ is increasing from zero in the interval $q_m \leqslant x \leqslant 1$, (2.44) holds for $0 \leqslant x \leqslant 1$, $m \leqslant n-2$. When $m = n-1$, $Q^{(m)}(x)$ is constant and also then (2.44) holds.

(c) From (2.46) and (2.1) we have

$$Q'(x) = \left(1 - \frac{1}{n}\right) T_n'(x) + \frac{1}{n} (1-x)\, T_n''(x)$$
$$= \frac{n-1}{n} \left(\left(1 + \frac{1}{n-1} \frac{x}{1+x}\right) T_n'(x) - \frac{n^2}{n-1} \frac{1}{1+x}\, T_n(x) \right).$$

Using inequality (2.11) we obtain

$$|Q'(x)| \leqslant (n-1)\, n\left(\left(1+\frac{1}{n-1}\frac{x}{1+x}\right)\frac{1}{n\sqrt{1-x^2}}+\frac{1}{n-1}\frac{1}{1+x}\right).$$

When $n \geqslant 10$, we see that

$$|Q'(x)| \leqslant Q'(1)\left(\left(1+\frac{1}{9}\frac{x}{1+x}\right)\frac{1}{10}\frac{1}{\sqrt{1-x^2}}+\frac{1}{9}\frac{1}{1+x}\right)$$
$$\leqslant \frac{1}{3}\, Q'(1) \qquad \text{if} \quad 0 \leqslant x \leqslant 0.9.$$

When $x=\omega_1$ and $n \geqslant 10$,

$$|Q'(\omega_1)| = \frac{1-\omega_1}{n}\,|T_n''(\omega_1)| = \frac{1}{n(1+\omega_1)}\, n^2 \leqslant \frac{1}{3}\, Q'(1).$$

In the remaining case, $0.9 \leqslant x \leqslant \omega_1$, we only need to study $Q'(x)$ for those points x, where $Q''(x)=0$. With $m=2$, (2.46) and (2.1) yield

$$Q''(x) = \left(1-\frac{2}{n}\right) T_n''(x) + \frac{1}{n}\,(1-x)\, T_n'''(x)$$
$$= \left(1-\frac{2}{n}+\frac{3x}{n(1+x)}\right) T_n''(x) - \frac{n^2-1}{n(1+x)}\, T_n'(x) = 0,$$

if

$$T_n''(x) = \frac{n^2-1}{n(1+x)-2(1+x)+3x}\, T_n'(x).$$

This gives, using the estimate mentioned in Remark 2.1,

$$|Q'(x)| \leqslant \frac{n-1}{n}\,|T_n'(x)|\left(1+(1-x)\,\frac{n+1}{n(1+x)+x-2}\right)$$
$$\leqslant n(n-1)\,\frac{1}{3}\frac{8}{9}\left(1+\frac{2}{3}\,(1-x)\right) \leqslant n(n-1)/3,$$

if $0.9 \leqslant x \leqslant \omega_1$, where we also used the inequality $(n+1)/(n(1+x)+x-2) \leqslant 2/3$, which holds for $n \geqslant 10$ and $x \geqslant 0.9$.

This completes the proof of Lemma 4.

3. PROOFS OF THE THEOREMS

In this section we prove Theorems 1 and 3 for $2 \leqslant m \leqslant n-1$, leaving the case $m=1$ to Section 4.

In Theorem 3, we will apply the Lagrange interpolation formula to the polynomials in D_{n-1} with y_j, $0 \leqslant j \leqslant n-1$, as interpolation points. With

$$\omega(x) = (x-1)\, T'_n(x) = L_n(x), \tag{3.1}$$

we see from (2.15), that for $p \in D_{n-1}$, the Lagrange interpolation formula gives

$$p(x) = \sum_{j=0}^{n-1} \frac{p(y_j)}{\omega'(y_j)} \frac{\omega(x)}{x - y_j} = \frac{p(1)}{n^2} T'_n(x) + \sum_{j=1}^{n-1} p(y_j) \frac{1 + y_j}{n^2} (-1)^j L_{n,j}(x). \tag{3.2}$$

The derivative of (3.2) of order m is

$$p^{(m)}(x) = \frac{p(1)}{n^2} T_n^{(m+1)}(x) + \sum_{j=1}^{n-1} p(y_j) \frac{1 + y_j}{n^2} (-1)^j L_{n,j}^{(m)}(x). \tag{3.3}$$

From (3.3), for $p \in D_{n-1}$, we obtain the inequality, sharp for every x,

$$|p^{(m)}(x)| \leqslant \frac{|T_n^{(m+1)}(x)|}{n^2} + \sum_{j=1}^{n-1} \frac{1 + y_j}{n^2} |L_{n,j}^{(m)}(x)| = A_m(x), \tag{3.4}$$

where $A_m(x)$ is independent of p.

We start by proving that the inequality $|p^{(m)}(x)| \leqslant |Q^{(m)}(x)|$ holds in some subintervals of $[-1, 1]$, $1 \leqslant m \leqslant n-1$.

Let $\omega_{m+1,1} > \omega_{m+1,2} > \cdots > \omega_{m+1,n-m-1}$, be the zeros of $T_n^{(m+1)}(x)$ and let $\bar{\mu}_{j,1} > \bar{\mu}_{j,2} > \cdots > \bar{\mu}_{j,n-m-1}$ be the zeros of $L_{n,j}^{(m)}(x)$, $1 \leqslant j \leqslant n-1$. Then we have from (2.17)

$$\begin{aligned} \bar{\mu}_{n-1,1} &> \bar{\mu}_{n-2,1} > \cdots > \bar{\mu}_{1,1} > \omega_{m+1,1} \\ &> \bar{\mu}_{n-1,2} > \bar{\mu}_{n-2,2} > \cdots > \bar{\mu}_{1,2} > \omega_{m+1,2} \\ &\;\vdots \\ &> \bar{\mu}_{n-1,n-m-1} > \bar{\mu}_{n-2,n-m-1} > \cdots > \bar{\mu}_{1,n-m-1} > \omega_{m+1,n-m-1}. \end{aligned}$$

In the intervals

$$I_1 = [\bar{\mu}_{n-1,1}, 1], \qquad I_2 = [\bar{\mu}_{n-1,2}, \omega_{m+1,1}],$$

$$I_3 = [\bar{\mu}_{n-1,3}, \omega_{m+1,2}], ..., I_{n-m} = [-1, \omega_{m+1,n-m-1}],$$

all the functions $T_n^{(m+1)}$, $L_{n,1}^{(m)}, ..., L_{n,n-1}^{(m)}$, have the same sign, and thus (3.3) with $p=Q$ gives

$$|Q^{(m)}(x)| = \left|\frac{1}{n^2}\, T_n^{(m+1)}(x) + \sum_{j=1}^{n-1} \frac{1+y_j}{n^2}\, L_{n,j}^{(m)}(x)\right|$$
$$= \frac{1}{n^2}\, |T_n^{(m+1)}(x)| + \sum_{j=1}^{n-1} \frac{1+y_j}{n^2}\, |L_{n,j}^{(m)}(x)|.$$

If x belongs to one of these intervals, and if $p \in D_{n-1}$, we have

$$|p^{(m)}(x)| = \left|p(1)\, \frac{T_n^{(m+1)}(x)}{n^2} + \sum_{j=1}^{n-1} p(y_j)\, \frac{1+y_j}{n^2}\, (-1)^j\, L_{n,j}^{(m)}(x)\right|$$
$$\leqslant \frac{|T_n^{(m+1)}(x)|}{n^2} + \sum_{j=1}^{n-1} \frac{1+y_j}{n^2}\, |L_{n,j}^{(m)}(x)| = |Q^{(m)}(x)|, \tag{3.5}$$

where the inequality is strict when x is an interior point of some I_k, unless $p(x) = \pm Q(x)$. If moreover $0 \leqslant x \leqslant 1$, we have by (2.44), for $1 \leqslant m \leqslant n-1$,

$$|p^{(m)}(x)| \leqslant |Q^{(m)}(x)| \leqslant Q^{(m)}(1), \tag{3.6}$$

i.e., inequality (1.15) of Theorem 3 holds for $x \geqslant 0$ in the above mentioned intervals.

Proof of Theorem 3. We divide the proof into three cases. In Case 1 we consider $m=n-1$ and $m=n-2$; in Cases 2 and 3, with $2 \leqslant m \leqslant n-3$, we study the intervals $\omega_m \leqslant x \leqslant 1$ and $0 \leqslant x \leqslant \omega_m$, respectively.

Case 1. $n-2 \leqslant m \leqslant n-1$ and $0 \leqslant x \leqslant 1$.

When $m=n-1$, $p^{(n-1)}(x)$ is a constant, and thus the inequality $|p^{(n-1)}(x)| = |p^{(n-1)}(1)| \leqslant Q^{(n-1)}(1)$ follows from (3.5).

When $m=n-2$, $p^{(n-2)}(x)$ is linear,

$$1 \in I_1 = [(1+\cos(\pi/n))/(n-1), 1], \qquad 0 \in I_2 = [-1, 0],$$

and (3.6) gives

$$|p^{(n-2)}(1)| \leqslant Q^{(n-2)}(1) \qquad \text{and} \qquad |p^{(n-2)}(0)| \leqslant |Q^{(n-2)}(0)| \leqslant Q^{(n-2)}(1),$$

i.e., (1.15) holds.

Case 2.1 $2 \leqslant m \leqslant n-3$, $n \geqslant 6$, and $\omega_m \leqslant x \leqslant 1$.

When $x \geqslant \mu_{n-1,m}$, we already know that $|p^{(m)}(x)| \leqslant Q^{(m)}(1)$, also for $m=1$, according to (3.6), so we restrict ourselves to the interval $\omega_m \leqslant x \leqslant \mu_{n-1,m}$, and suppose that $2 \leqslant m \leqslant n-3$. We have $q_m < \omega_m < \lambda_m$ and $\mu_{n-1,m} < \lambda_m$ from Lemma 1 and moreover we have $Q^{(m)}(x) > 0$ for $x > q_m$.

Let $A_+(x)$ be the sum of the positive terms and $A_-(x)$ be the absolute value of the sum of the negative terms in the right-hand side of

$$Q^{(m)}(x) = \frac{1}{n^2} T_n^{(m+1)}(x) + \sum_{j=1}^{n-1} \frac{1+y_j}{n^2} L_{n,j}^{(m)}(x).$$

Since $Q^{(m)}(x) = A_+(x) - A_-(x) \geqslant 0$, we see that $A_+(x) \geqslant A_-(x)$ for $x \geqslant q_m$ and, since $A_+(x)$ is increasing, the inequality

$$\begin{aligned} |p^{(m)}(x)| &\leqslant A_+(x) + A_-(x) \leqslant 2A_+(x) \leqslant 2A_+(\lambda_m) = 2Q^{(m)}(\lambda_m) \\ &= 2T_n^{(m)}(\lambda_m) \leqslant (1-m/n)\, T_n^{(m)}(1), \end{aligned} \tag{3.7}$$

will follow as soon as we show that

$$T_n^{(m)}(\lambda_m) \leqslant (1-m/n)\, T_n^{(m)}(1)/2. \tag{3.8}$$

By the definition of λ_m,

$$L_n^{(m)}(\lambda_m) = (\lambda_m - 1)\, T_n^{(m+1)}(\lambda_m) + mT_n^{(m)}(\lambda_m) = 0,$$

i.e.,

$$(1-\lambda_m)\, T_n^{(m+1)}(\lambda_m) = mT_n^{(m)}(\lambda_m).$$

Since $\lambda_m > \omega_m$, we have $T_n^{(m+r)}(\lambda_m) \geqslant 0$, $r \geqslant 0$, and according to the Taylor expansion

$$\begin{aligned} T_n^{(m)}(1) &\geqslant T_n^{(m)}(\lambda_m) + (1-\lambda_m)\, T_n^{(m+1)}(\lambda_m) \\ &= (m+1)\, T_n^{(m)}(\lambda_m). \end{aligned} \tag{3.9}$$

From (3.9) we see that

$$T_n^{(m)}(\lambda_m) \leqslant T_n^{(m)}(1)/(m+1) \leqslant (1-m/n)\, T_n^{(m)}(1)/2,$$

giving (3.8), if

$$\frac{1}{m+1} \leqslant \frac{1}{2}\frac{n-m}{n} \qquad \text{or} \qquad \frac{2}{m+1} + \frac{m}{n} \leqslant 1.$$

In the last inequality, the left-hand side is convex as a function of m. The inequality holds for $m=2$ and $m=n-3$ when $n \geqslant 6$, and thus for $2 \leqslant m \leqslant n-3$. This proves (1.15) for $\omega_m \leqslant x \leqslant 1$, $2 \leqslant m \leqslant n-3$, $n \geqslant 6$.

Case 2.2. $(n, m) = (5, 2)$ and $\omega_2 \leqslant x \leqslant \lambda_2$.

Here $L_5''(x) = 1600x^3 - 960x^2 - 360x + 120$. Since $\omega_2 < \lambda_2 \leqslant 0.77$, we get (3.8) from

$$T_5''(\lambda_2) \leqslant T_5''(0.77) < 54 < 60 = 0.5(1 - 2/5)\, T_5''(1).$$

Case 3.1. $2 \leqslant m \leqslant n-3$, $0 \leqslant x \leqslant \omega_m$, and $U(x) \geqslant V(x)$.

If $p \in D_{n-1}$, and if we choose the constant A, such that $P \in C_n$, where

$$P(x) = p(x) + AL_n(x),$$

the Duffin and Schaeffer theorem [D&S] for (1.14), with $m = n$ yields

$$|A|\, n2^{n-1}n! \leqslant 2^{n-1}n! \qquad \text{or} \qquad |A| \leqslant 1/n.$$

Theorem B implies that $|P^{(m)}(x)| \leqslant \max\{U(x), V(x)\}$, where

$$U(x) = |T_n^{(m)}(x)| \leqslant T_n^{(m)}(1)/(2m+1), \qquad 0 \leqslant x \leqslant \omega_m,$$

according to Lemma 2, and, if we also use (2.1) and (2.3),

$$\begin{aligned} V(x) &= \frac{1}{m}\,|(1-x^2)\, T_n^{(m+1)}(x) - mxT_n^{(m)}(x)| \\ &= \frac{1}{m}\,|(m-1)\, xT_n^{(m)}(x) - (n^2 - (m-1)^2)\, T_n^{(m-1)}(x)| \\ &\leqslant \frac{m-1}{m}\, x\, \frac{T_n^{(m)}(1)}{2m+1} + \frac{n^2 - (m-1)^2}{m}\, \frac{T_n^{(m-1)}(1)}{2m-1} \\ &= \left(\frac{m-1}{m}\, \frac{x}{2m+1} + \frac{1}{m}\right) T_n^{(m)}(1), \qquad 0 \leqslant x \leqslant \omega_m. \end{aligned} \tag{3.10}$$

When $U(x) \geqslant V(x)$, we obtain, also using (2.36),

$$\begin{aligned} |p^{(m)}(x)| &\leqslant U(x) + |L_n^{(m)}(x)|/n \leqslant \left(\frac{1}{2m+1} + \frac{1}{n}\left(1 + \frac{m}{2m+1}\right)\right) T_n^{(m)}(1) \\ &\leqslant \frac{n-m}{n}\, T_n^{(m)}(1) = Q^{(m)}(1), \end{aligned}$$

if

$$\frac{1}{2m+1} + \frac{1}{n}\left(1 + \frac{m}{2m+1}\right) \leqslant \frac{n-m}{n}$$

or, after simplification,

$$2 + 1/(2m) \leqslant n - m,$$

which holds for all m, $2 \leqslant m \leqslant n-3$, $n \geqslant 5$.

Case 3.2. $2 \leqslant m \leqslant n-4$, $0 \leqslant x \leqslant \omega_m$, and $V(x) \geqslant U(x)$.
When $V(x) \geqslant U(x)$, we instead obtain from (3.10) and (2.36),

$$\begin{aligned} |p^{(m)}(x)| &\leqslant V(x) + |L_n^{(m)}(x)|/n \\ &\leqslant \left(\frac{m-1}{m} \frac{1}{2m+1} + \frac{1}{m} + \frac{1}{n} \left(1 + \frac{m}{2m+1} \right) \right) T_n^{(m)}(1) \\ &\leqslant \frac{n-m}{n} T_n^{(m)}(1), \end{aligned} \tag{3.11}$$

where the last inequality holds if

$$m + 3 + 7/(2m-2) \leqslant n. \tag{3.12}$$

We see that (3.12) holds for $m = 2$ if $n \geqslant 9$, for $m = 3$ if $n \geqslant 8$, for $m = 4$ if $n \geqslant 9$ and for $m \geqslant 5$ if $m \leqslant n-4$.

Case 3.3. $m = n-3$, $0 \leqslant x \leqslant \omega_m$, and $V(x) \geqslant U(x)$.
When $m = n-3$,

$$\begin{aligned} T_n^{(n-3)}(x) &= 2^{n-1} n!(x^3 - 1.5x/(n-1))/6, \\ T_n^{(n-2)}(x) &= 2^{n-1} n!(3x^2 - 1.5/(n-1))/6 \end{aligned}$$

and

$$\begin{aligned} V(x) = \frac{2^{n-1} n!}{6} \bigg| & \frac{1.5}{(n-1)(n-3)} - \left(\frac{3}{n-3} + \frac{1.5}{(n-1)(n-3)} \right) x^2 \\ & + \left(\frac{3}{n-3} + 1 \right) x^4 \bigg|. \end{aligned}$$

In the interval $0 \leqslant x \leqslant \omega_{n-3} = \sqrt{1.5/(n-1)}$ we have

$$V(x) \leqslant V(0) = \frac{2^{n-1} n!}{6} \frac{1.5}{(n-1)(n-3)} = \frac{1.5}{(n-2.5)(n-3)} T_n^{(n-3)}(1).$$

Using this inequality and (2.36) we see that

$$|p^{(n-3)}(x)| \leqslant V(x) + \frac{1}{n} |L_n^{(n-3)}(x)|$$

$$\leqslant \left(\frac{1.5}{(n-2.5)(n-3)} + \frac{1.5}{n} \right) T_n^{(n-3}(1) \leqslant \frac{3}{n} T_n^{(n-3)}(1),$$

when $0 \leqslant x \leqslant \omega_{n-3}$, $n \geqslant 5$.

Case 3.4. The remaining cases with $0 \leqslant x \leqslant \omega_m$ and $V(x) \geqslant U(x)$ are $(n, m) = (6, 2)$, $(7, 2)$, $(8, 2)$, $(7, 3)$, and (8.4).

If $\omega_{m,n} = \omega_m$ is the largest zero of $T_n^{(m)}(x)$, it is easy to see that $\omega_{2,6} < 0.8$, $\omega_{2,7} < 0.8$, $\omega_{2,8} < 0.9$, $\omega_{3,7} < 0.7$, and $\omega_{4,8} < 0.7$. According to (3.10) and (3.11) it is enough to prove that

$$V(x) + \frac{1}{n} |L_n^{(m)}(x)| \leqslant \frac{n-m}{n} T_n^{(m)}(1), \qquad 0 \leqslant x \leqslant \omega_{m,n}.$$

We write this inequality in each case for a little larger interval, containing $\omega_{m,n}$,

$$4 \cdot 3 \, |-240x^5 + 256x^3 - 51x|$$
$$+ 4 \, |240x^4 - 160x^3 - 96x^2 + 48x + 3| \leqslant 4 \cdot 70, \qquad 0 \leqslant x \leqslant 0.8,$$

$$8 \cdot 7 \, |-168x^6 + 220x^4 - 69x^2 + 3| + 8 \cdot 2 \, |168x^5 - 120x^4$$
$$- 100x^3 + 60x^2 + 9x - 3| \leqslant 8 \cdot 70, \qquad 0 \leqslant x \leqslant 0.8,$$

$$16 \cdot 4 \, |-448x^7 + 696x^5 - 300x^3 + 31x| + 16 \, |448x^6 - 336x^5$$
$$- 360x^4 + 240x^3 + 60x^2 - 30x - 1| \leqslant 16 \cdot 63, \qquad 0 \leqslant x \leqslant 0.9,$$

$$16 \cdot 7 \, |-280x^5 + 260x^3 - 43x|$$
$$+ 16 \cdot 3 \, |280x^4 - 160x^3 - 100x^2 + 40x + 3| \leqslant 16 \cdot 252, \qquad 0 \leqslant x \leqslant 0.7,$$

and

$$1920 \cdot 2 \, |-112x^5 + 92x^3 - 13x|$$
$$+ 1920 \, |112x^4 - 56x^3 - 36x^2 + 12x + 1| \leqslant 1920 \cdot 33, \qquad 0 \leqslant x \leqslant 0.7.$$

Elementary calculations show that these inequalities hold and thus that (3.11) holds in all five remaining cases. This completes the proof of Theorem 3, when $2 \leqslant m \leqslant n-1$.

Proof of Theorem 1. First we suppose that $\varphi \in W_\infty^n[-\cos(\pi/n), 1]$, $n \geqslant 2$. We denote the sup norm of a function φ defined on the interval

$[-\cos(\pi/n), 1]$ by $\|\varphi\|_n$. Let p_{n-1} be the polynomial of degree $\leqslant n-1$ that interpolates φ at the points $y_k = \cos(k\pi/n)$, $0 \leqslant k \leqslant n-1$. We write

$$\varphi(y) = p_{n-1}(y) + R_n(y) \qquad \text{and} \qquad \varphi^{(m)}(y) = p^{(m)}_{n-1}(y) + R^{(m)}_n(y).$$

Here we use Theorem 3 and Theorem A in order to estimate $p^{(m)}_{n-1}(y)$ and $R^{(m)}_n(y)$, respectively.

According to Theorem 3, we have for $2 \leqslant m \leqslant n-1$,

$$|p^{(m)}_{n-1}(y)| \leqslant \|\varphi\|_n (1 - m/n)\, T^{(m)}_n(1), \qquad 0 \leqslant y \leqslant 1. \tag{3.13}$$

With interpolation points y_k, $0 \leqslant k \leqslant n-1$, we have in Theorem A,

$$\omega(x) = \frac{1}{2^{n-1}n} L_n(x), \qquad \omega_0(x) = \frac{1}{2^{n-1}n} T'_n(x), \qquad \text{and}$$

$$\omega_{n-1}(x) = \frac{1}{2^{n-1}n} L_{n,n-1}(x).$$

Hence α_j and β_j, $1 \leqslant j \leqslant n-1-m$, are the zeros of $T^{(m+1)}_n(x)$ and $L^{(m)}_{n,n-1}(x)$, respectively. Thus Theorem A, (2.6), (2.37), and (2.38) imply that

$$\sup_{w \leqslant x \leqslant 1} |\varphi^{(m)}(x) - p^{(m)}_{n-1}(x)| \leqslant \frac{\|\varphi^{(n)}\|_n}{n!} \sup_{w \leqslant x \leqslant 1} \frac{|L^{(m)}_n(x)|}{2^{n-1}n}$$
$$= \frac{\|\varphi^{(n)}\|_n}{n!} \frac{|L^{(m)}_n(1)|}{2^{n-1}n},$$

if $w = 0 \in I_{n,m}$, when $n-m$ is even, and if $w = -\omega^+_{m+1} \in I_{n,m}$, when $n-m$ is odd, $2 \leqslant m \leqslant n-3$. Hence, Theorem A in both cases gives

$$|R^{(m)}_n(y)| \leqslant \frac{\|\varphi^{(n)}\|_n}{n!} \frac{mT^{(m)}_n(1)}{2^{n-1}n}, \qquad 0 \leqslant y \leqslant 1. \tag{3.14}$$

Thus, using both (3.13) and (3.14) we obtain

$$|\varphi^{(m)}(y)| \leqslant \|\varphi\|_n (1 - m/n)\, T^{(m)}_n(1) + \frac{\|\varphi^{(n)}\|_n}{n!} \frac{m}{n} \frac{T^{(m)}_n(1)}{2^{n-1}}, \qquad 0 \leqslant y \leqslant 1 \tag{3.15}$$

for $2 \leqslant m \leqslant n-2$.

In order to prove (3.14) when $m=n-1$, we use divided differences and obtain

$$d_{n-1}[\varphi; u_0, ..., u_{n-1}] - d_{n-1}[\varphi; y_0, ..., y_{n-1}]$$
$$= \sum_{j=0}^{n-1} (u_j - y_j)\, d_n[\varphi; y_0, ..., y_j, u_j, ..., u_{n-1}].$$

Letting $u_0, ..., u_{n-1}$ tend to y, $0 \leqslant y \leqslant 1$, we see that

$$\frac{|\varphi^{(n-1)}(y) - p_{n-1}^{(n-1)}(y)|}{(n-1)!} \leqslant \sum_{j=0}^{n-1} |y - y_j| \frac{\|\varphi^{(n)}\|_n}{n!}.$$

Here $g(y) = \sum_{j=0}^{n-1} |y - y_j|$ is convex, with $g(1) = n-1$, $g(0) \leqslant n-1$. Hence $g(y) \leqslant n-1$, $0 \leqslant y \leqslant 1$, and

$$|R_n^{(n-1)}(y)| = |\varphi^{(n-1)}(y) - p_{n-1}^{(n-1)}(y)| \leqslant \frac{n-1}{n} \|\varphi^{(n)}\|_n, \qquad 0 \leqslant y \leqslant 1,$$

which is (3.14) when $m=n-1$, and thus (3.15) holds also for $m=n-1$.

Finally, suppose that $f \in W_\infty^n[-1, 1]$. Then we define an auxiliary function φ, in the following way.

If $0 < c \leqslant 1$, we set

$$\varphi(y) = f(x - cx + cy), \qquad \text{where} \quad 0 \leqslant x \leqslant 1,$$

and if $1 < c \leqslant 2/(1+\cos(\pi/n)) = (\cos(\pi/2n))^{-2}$, we define

$$\varphi(y) = f(1 - c + cy),$$

where $y \in [-\cos(\pi/n), 1]$ in both cases.

With $\alpha = x - cx$ and $\alpha = 1 - c$, respectively, we have in both cases $-1 \leqslant \alpha + cy \leqslant 1$ for all values of c, x, and y considered. This yields $|\varphi(y_k)| \leqslant \|f\|$, $0 \leqslant k \leqslant n-1$, and $\varphi^{(m)}(y) = c^m f^{(m)}(\alpha + cy)$, $1 \leqslant m \leqslant n$. Since

$$\|\varphi^{(m)}\|_n \leqslant c^m \|f^{(m)}\|, \qquad 0 \leqslant m \leqslant n,$$

(3.15) gives

$$|c^m f^{(m)}(\alpha + cy)|$$
$$\leqslant \|f\| (1 - m/n)\, T_n^{(m)}(1) + c^n \frac{\|f^{(n)}\|}{n!} \frac{m}{2^{n-1} n} T_n^{(m)}(1), \qquad 0 \leqslant y \leqslant 1. \tag{3.16}$$

We now write $\alpha + cy = x$ in both cases, for $0 \leqslant x \leqslant 1$. These values of x correspond to $0 \leqslant y \leqslant 1$, and $0 < 1 - 1/c \leqslant y \leqslant 1$, respectively, and thus we obtain

$$|c^m f^{(m)}(x)| \leqslant \|f\| \, (1 - m/n) \, T_n^{(m)}(1) + c^n \frac{\|f^{(n)}\|}{n!} \frac{m}{2^{n-1}n} T_n^{(m)}(1),$$
$$0 \leqslant x \leqslant 1. \tag{3.17}$$

Hence Theorem 1 is proved, after considering also $f(-x)$, $0 \leqslant x \leqslant 1$, and the proofs of the theorems are completed for $2 \leqslant m \leqslant n - 1$.

4. THE CASE $m = 1$

Also in this case we start with Theorem 3, and show that $|p'(x)| \leqslant Q'(1) = n(n-1)$, $0 \leqslant x \leqslant 1$, when $p \in D_{n-1}$. The cases $n = 2$ and 3 were settled in Section 3. When $4 \leqslant n \leqslant 9$, we rely on the result of the elementary calculations, which can be made after inserting the derivatives of

$$L_{n,j}(x) = 2^{n-1} n(x-1) \cdots (x - y_{j-1})(x - y_{j+1}) \cdots (x - y_{n-1}),$$
$$1 \leqslant j \leqslant n-1,$$

into (3.4), and which give $A_1(x) \leqslant A_1(1)$, $0 \leqslant x \leqslant 1$. Thus

$$|p'(x)| \leqslant A_1(x) \leqslant A_1(1) = Q'(1), \qquad 0 \leqslant x \leqslant 1, 4 \leqslant n \leqslant 9.$$

From now on we suppose that $n \geqslant 10$, and start with the interval $\omega_1 \leqslant x \leqslant 1$. According to (3.6) it is enough to consider the interval $\omega_1 \leqslant x \leqslant \mu_{n-1,1}$ or the somewhat larger interval $\omega_1 \leqslant x \leqslant \lambda_1$. Since $\omega_1 = \cos(\pi/n) < \lambda_1 = \cos \alpha$, $\alpha = \alpha_n$, we have $0 < \alpha < \pi/n$. From

$$\begin{aligned} 0 = L_n'(\lambda_1) &= (T_n'(\lambda_1) + n^2 T_n(\lambda_1))/(1 + \lambda_1) \\ &= (n \sin(n\alpha)/\sin \alpha + n^2 \cos(n\alpha))/(1 + \lambda_1) \end{aligned}$$

we obtain $\tan(n\alpha) = -n \sin \alpha$, and deduce that $\alpha = (2\pi/3 - k_n)/n$, where k_n is increasing and $0.063 \leqslant k_n \leqslant 0.066$ for $n \geqslant 10$. Thus

$$\begin{aligned} T_n'(\lambda_1) = -n^2 T_n(\lambda_1) &= -n^2 \cos(2\pi/3 - k_n) \\ &= n^2 \cos(\pi/3 + k_n) \leqslant n^2 \cos(\pi/3 + 0.063) \leqslant 0.45 n^2. \end{aligned}$$

Using this result in (3.7) we obtain

$$|p'(x)| \leqslant 2T_n'(\lambda_1) \leqslant 0.9n^2 \leqslant n(n-1), \qquad \omega_1 \leqslant x \leqslant \lambda_1, n \geqslant 10.$$

It still remains to prove that $|p'(x)| \leqslant Q'(1)$, $0 \leqslant x \leqslant \omega_1$ when $n \geqslant 10$. We suppose that $p \neq \pm Q$, since (2.44) gives $|Q'(x)| \leqslant Q'(1)$. In the proof we will use the inequalities

$$|Q'(y_k)| \leqslant n, \qquad 1 \leqslant k \leqslant n/2 \tag{4.1}$$

$$|Q'(x)| \leqslant Q'(1)/3, \qquad 0 \leqslant x \leqslant \omega_1, n \geqslant 10, \tag{4.2}$$

and

$$|p'(y_k)| \leqslant 0.5n^2, \qquad 1 \leqslant k \leqslant n/2, n \geqslant 10, \tag{4.3}$$

where (4.2) was proved in Lemma 4.

Moreover, we will prove that the functions $Q'(x) \pm p'(x)$ both have exactly one local minimum (maximum) in the interval $[y_{k+1}, y_k]$, $y_{k+1} \geqslant 0$, when k is odd (even). Also, if n is odd, and if each of the functions $Q'(x) \pm p'(x)$ has a local extreme value in the interval $[0, y_r]$, $r = (n-1)/2$, we will prove that they are both of the same type. From (4.1) and (4.3), we obtain for $1 \leqslant k \leqslant n/2$,

$$|Q'(y_k) \pm p'(y_k)| \leqslant n + 0.5n^2 \leqslant 2n(n-1)/3 = 2Q'(1)/3, \qquad n \geqslant 10. \tag{4.4}$$

When n is even, one $y_k = 0$, and when n is odd, $T_n''(0) = 0$. In the latter case $|p'(0)| \leqslant |Q'(0)|$ according to the proof of (3.6), and then (4.2) yields

$$|Q'(0) \pm p'(0)| \leqslant 2\,|Q'(0)| \leqslant 2Q'(1)/3, \tag{4.5}$$

and thus $|Q'(0) \pm p'(0)| \leqslant 2Q'(1)/3$ holds for all values of n.

We now prove that $|p'(x)| \leqslant Q'(1)$ by studying the converse inequality. If $|p'(x_0)| > Q'(1)$ for some x_0, $0 < x_0 < y_1$, and if $0 \leqslant y_{p+1} \leqslant x_0 \leqslant y_p$ (or $0 < x_0 < y_r$, $r = (n-1)/2$, when n is odd), then $|Q'(x_0) \pm p'(x_0)| > 2Q'(1)/3$, according to (4.2), and $|Q'(x) \pm p'(x)| \leqslant 2Q'(1)/3$ at the endpoints of the interval $[y_{p+1}, y_p]$ or $[0, y_r]$ according to (4.4) or (4.5). Then one of the functions $Q'(x) \pm p'(x)$ attains a local maximum and the other attains a local minimum in the interval mentioned. But this is impossible according to what was said above about local extreme points of $Q'(x) \pm p'(x)$. Thus $|p'(x)| \leqslant Q'(1)$, $0 \leqslant x \leqslant 1$.

Last, we will below in (I)–(III) prove (4.1), the assertion about the local extreme points of $Q'(x) \pm p'(x)$ and (4.3).

(I) When $1 \leqslant k \leqslant n/2$, we see from (2.46) that (4.1) holds, since

$$|Q'(y_k)| = \left| \frac{1 - y_k}{n} T_n''(y_k) \right| = \frac{|n^2 T_n(y_k)|}{n(1 + y_k)} \leqslant n.$$

(II) Next we prove the assertion about the local extreme points of $Q'(x) \pm p'(x)$. Denote the zeros of $T_n^{(3)}(x)$ and $L''_{n,n-1}(x)$ by $\omega_{3,k}$ and $\bar{\mu}_{n-1,k}$, respectively, $1 \leqslant k \leqslant n-3$. From (2.9) and the Sturm comparison theorem we see that if $p > m$, then $S_p(x) < S_m(x)$, and thus $T_n^{(m)}(x)$ always has a zero between any two zeros of $T_n^{(p)}(x)$. This gives, taking account of all zeros of T'_n and $T_n^{(3)}$, $0 \leqslant y_r \leqslant \omega_{3,r-1} < y_{r-1} < \omega_{3,r-2} < \cdots < y_3 < \omega_{3,2} < y_2 < \omega_{3,1} < y_1$ where

$$0 < y_r < \omega_{3,r-1}, \qquad \text{if} \quad n = 2r+1,$$

and

$$0 = y_r = \omega_{3,r-1}, \qquad \text{if} \quad n = 2r.$$

From

$$(x + \cos(\pi/n))\, L'_{n,n-1}(x) + L_{n,n-1}(x) = (x-1)\, T''_n(x) + T'_n(x)$$

which gives

$$(x + \cos(\pi/n))\, L''_{n,n-1}(x) + 2L'_{n,n-1}(x) = \frac{(2-x)\, T''_n(x) + (n^2-1)\, T'_n(x)}{x+1}, \tag{4.6}$$

and observing that $\operatorname{sgn}(T''_n(y_k)) = (-1)^{k+1}$, we obtain for $1 \leqslant k \leqslant n-2$,

$$\operatorname{sgn}(L'_{n,n-1}(y_k)) = (-1)^k \qquad \text{and} \qquad \operatorname{sgn}(L''_{n,n-1}(y_k)) = (-1)^{k+1}.$$

Thus $L''_{n,n-1}(x)$ has a zero in each interval, $[y_{k+1}, y_k]$, $1 \leqslant k \leqslant n-3$, and since $L''_{n,n-1}(x)$ has exactly $n-3$ zeros, we see that

$$y_{k+1} < \bar{\mu}_{n-1,k} < y_k, \qquad 1 \leqslant k \leqslant n-3.$$

From (2.17) we obtain

$$\cdots < \omega_{3,3} < \bar{\mu}_{n-1,3} < \omega_{3,2} < \bar{\mu}_{n-1,2} < \omega_{3,1} < \bar{\mu}_{n-1,1}.$$

Combining the above three series of inequalities we see that for $y_k \geqslant 0$,

$$\cdots < \omega_{3,3} < \bar{\mu}_{n-1,3} < y_3 < \omega_{3,2} < \bar{\mu}_{n-1,2} < y_2 < \omega_{3,1} < \bar{\mu}_{n-1,1} < y_1$$

which implies that

$$y_k \in [\bar{\mu}_{n-1,k}, \omega_{3,k-1}], \qquad k \geqslant 2,\ y_k \geqslant 0.$$

If $n = 2r$, $0 = y_r \in [\bar{\mu}_{n-1,r}, \omega_{3,r-1}] = [\bar{\mu}_{n-1,r}, 0]$.

Since $p(x) \neq \pm Q(x)$, we have from the proof of (3.6), $|p''(x)| < |Q''(x)|$ in the interior of the intervals $[\bar{\mu}_{n-1,k}\omega_{3,k-1}]$, $2 \leqslant k \leqslant n-3$, for $x > \bar{\mu}_{n-1,1}$ and for $x < \omega_{3,n-3}$. Hence each of the functions $Q''(x) \pm p''(x)$ has at least one, i.e., exactly one zero in each interval $[\omega_{3,k}, \bar{\mu}_{n-1,k}]$, $1 \leqslant k \leqslant n-3$, and thus exactly one zero in each interval $[y_{k+1}, y_k]$, $y_{k+1} \geqslant 0$, $k \geqslant 1$. This means that each of the functions $Q'(x) \pm p'(x)$ has a local minimum at this zero when k is odd, and a local maximum, when k is even. When n is odd, $(n-1)/2 = r$, we have $y_{r+1} < 0 < y_r$, and

$$y_r \in (\bar{\mu}_{n-1,r}, \omega_{3,r-1}).$$

If $\bar{\mu}_{n-1,r} < 0$, then $Q''(x) \pm p''(x) \neq 0$ for $0 \leqslant x \leqslant y_r$. If $\bar{\mu}_{n-1,r} \geqslant 0$, both functions $Q'(x) \pm p'(x)$ can only have a local maximum or only have a local minimum in $[\omega_{3,r}, \bar{\mu}_{n-1,r}]$, and the same is of course true for the interval $[0, y_r]$, although one or both of $Q'(x) \pm p'(x)$ can fail to have a local extreme point there.

(III) Now it remains to prove (4.3). Using

$$\left|\frac{1+y_j}{n^2} L'_{n,j}(x)\right| = \left|\frac{1+y_j}{n^2}\left(D\left(\frac{x-1}{x-y_j}\right) T'_n(x) + \frac{x-1}{x-y_j} T''_n(x)\right)\right|$$

we obtain for $x = y_k$, $k \neq j$, $y_k \geqslant 0$,

$$\left|\frac{1+y_j}{n^2} L'_{n,j}(y_k)\right| = \left|\frac{1+y_j}{n^2} \frac{y_k-1}{y_k-y_j} T''_n(y_k)\right| = \left|\frac{1+y_j}{n^2} \frac{1}{y_k-y_j} \frac{n^2 T_n(y_k)}{1+y_k}\right|$$
$$= \frac{1}{|y_k-y_j|} + \frac{\operatorname{sgn}(k-j)}{1+y_k}. \tag{4.7}$$

Furthermore we have

$$\frac{|T''_n(y_k)|}{n^2} = \frac{|n^2 T_n(y_k)|}{n^2(1-y_k^2)} = \frac{1}{1-y_k^2}. \tag{4.8}$$

From (2.12) and (2.13) we obtain for $x = y_k$,

$$\left|\frac{1+y_k}{n^2} L'_{n,k}(y_k)\right| = \left|\frac{1+y_k}{n^2} \frac{1}{2} \frac{(2-y_k)\, T''_n(y_k)}{1+y_k}\right| = \left|\frac{1}{2n^2} \frac{(2-y_k)\, n^2 T_n(y_k)}{1-y_k^2}\right|$$
$$= \frac{2-y_k}{2} \frac{1}{1-y_k^2} = \frac{1}{2} \frac{1}{1-y_k^2} + \frac{1}{2} \frac{1}{1+y_k}. \tag{4.9}$$

Inserting (4.7)–(4.9) in (3.4), with $m=1$, we obtain

$$|p'(y_k)| \leqslant \frac{|T''_n(y_k)|}{n^2} + \sum_{j=1}^{n-1} \frac{1+y_j}{n^2} |L'_{n,j}(y_k)|$$

$$= \frac{1}{1-y_k^2} + \frac{1}{2}\frac{1}{1-y_k^2} + \frac{1}{2}\frac{1}{1+y_k} + \sum_{\substack{j=1\\ j\neq k}}^{n-1} \left(\frac{1}{|y_k - y_j|} + \frac{\operatorname{sgn}(k-j)}{1+y_k} \right)$$

$$= \frac{1.5}{1-y_k^2} + \sum_{\substack{j=1\\ j\neq k}}^{n-1} \frac{1}{|y_k - y_j|} - (n-2k-1/2)\frac{1}{1+y_k}. \tag{4.10}$$

As before we denote the right hand side of (4.10) by $A_1(y_k)$.
We start with the two single terms in $A_1(y_k)$,

$$U(y_k) = 1.5/(1-y_k^2) - (n-2k-1/2)/(1+y_k).$$

Using the inequality

$$\sin x \geqslant (\sin x_0)\, x/x_0, \qquad 0 < x \leqslant x_0 \leqslant \pi/2, \tag{4.11}$$

we obtain with $x_0 = \pi/6$,

$$U(y_1) \leqslant 1.5/\sin^2(\pi/n) \leqslant 1.5n^2/9 \leqslant 0.17n^2,$$

and with $x_0 = \pi/5$

$$U(y_k) < 1.5/\sin^2(2\pi/n) \leqslant 0.05n^2, \qquad 2 \leqslant k \leqslant n/6.$$

We also see that

$$U(y_k) \leqslant 3 \leqslant 0.03n^2, \qquad n \geqslant 10,\ n/6 < k \leqslant n/2.$$

Next we study the sum in $A_1(y_k)$ in the cases $1 \leqslant k \leqslant n/6$, $n/6 < k \leqslant n/3$, and $n/3 < k \leqslant n/2$.

Case 1. $1 \leqslant k \leqslant n/6$.

(a) When $j \leqslant n/3$,

$$(k+j)\,\pi/2n \leqslant \pi/4 \qquad \text{and} \qquad |k-j|\,\pi/2n \leqslant \pi/6.$$

Hence (4.11) gives

$$\begin{aligned} |y_k - y_j| &= |\cos(k\pi/n) - \cos(j\pi/n)| \\ &= 2\,|\sin((k-j)\,\pi/2n)\,\sin((k+j)\,\pi/2n)| \\ &\geqslant 2\,\frac{6}{\pi}\,\frac{1}{2}\,\frac{4}{\pi}\,\frac{\sqrt{2}}{2}\,|k^2 - j^2|\,\pi^2/(4n^2) \\ &= 3\sqrt{2}\,|k^2 - j^2|/n^2, \end{aligned} \tag{4.12}$$

and

$$\begin{aligned} S'_k &= \sum_{\substack{j=1\\ j\neq k}}^{n/3} \frac{1}{|y_k - y_j|} \leqslant \frac{n^2}{3\sqrt{2}} \left(\sum_{j=1}^{k-1} \frac{1}{k^2 - j^2} + \sum_{j=k+1}^{n-1} \frac{1}{j^2 - k^2} \right) \\ &\leqslant \frac{n^2}{3\sqrt{2}}\,\frac{1}{2k} \left(\sum_{j=1}^{k-1} \left(\frac{1}{k-j} + \frac{1}{k+j} \right) + \sum_{j=k+1}^{n-1} \left(\frac{1}{j-k} - \frac{1}{j+k} \right) \right) \\ &\leqslant \frac{n^2}{3\sqrt{2}}\,\frac{1}{2k}\,((1 + 1/2 + \cdots + 1/2k) + (1 + 1/2 + \cdots + 1/2k)). \end{aligned} \tag{4.13}$$

From the first two lines in (4.13) we see that

$$S'_1 \leqslant \frac{n^2}{6\sqrt{2}}\,(1 + 1/2) \leqslant 0.18n^2,$$

and from the last line that

$$S'_k \leqslant \frac{n^2}{6\sqrt{2}}\,(1 + 1/2 + 1/3 + 1/4) \leqslant 0.25n^2, \qquad 2 \leqslant k \leqslant n/6,$$

since the last line of (4.13) is the arithmetic mean of a decreasing sequence and hence decreasing.

(b) When $j > n/3$

$$\frac{1}{|\cos(k\pi/n) - \cos(j\pi/n)|} \leqslant \frac{1}{|\sqrt{3}/2 - 1/2|} = \sqrt{3} + 1 \leqslant 2.74,$$

which gives, if we improve the inequality a little for $k=1$,

$$S''_k = \sum_{j \geqslant n/3+1}^{n-1} \frac{1}{|y_k - y_j|} \leqslant 2.74\,\frac{2n}{3} \leqslant 0.19n^2 \qquad \text{and} \qquad S''_1 \leqslant 0.15n^2.$$

Thus

$$|p'(y_1)| \leqslant A_1(y_1) = U(y_1) + S'_1 + S''_1 \leqslant 0.17n^2 + 0.18n^2 + 0.15n^2 = 0.5n^2$$

and

$$\begin{aligned} |p'(y_k)| &\leqslant A_1(y_k) = U(y_k) + S'_k + S''_k \\ &\leqslant (0.05 + 0.25 + 0.19)\, n^2 < 0.5n^2, \qquad 2 \leqslant k \leqslant n/6. \end{aligned}$$

Hence (4.3) holds in Case 1.

Case 2. $n/6 < k \leqslant n/3$.

(a) When $1 \leqslant j \leqslant n/2$, we have $(k+j)\,\pi/2n \leqslant \pi/2$ and $|k-j|\,\pi/2n \leqslant \pi/6$. Hence

$$|y_k - y_j| \geqslant 2\,\frac{2}{\pi}\,\frac{6}{\pi}\,\frac{1}{2}\,\frac{|k^2 - j^2|\,\pi^2}{4n^2} = 3\,|k^2 - j^2|/n^2,$$

which compared to S'_k in Case 1 shows that the constant $3\sqrt{2}$ in (4.12) should be replaced by 3. This yields

$$S'_k = \sum_{\substack{j=1 \\ j \neq k}}^{n/2} \frac{1}{|y_k - y_j|} \leqslant \sqrt{2}\, 0.25n^2 \leqslant 0.36n^2.$$

(b) When $j \geqslant n/2$, we have $1/|y_k - y_j| \leqslant 2$, which gives

$$S''_k = \sum_{j \geqslant n/2}^{n-1} \frac{1}{|y_k - y_j|} \leqslant \frac{n}{2}\, 2 = n \leqslant 0.1n^2.$$

This shows that

$$|p'(y_k)| \leqslant A_1(y_k) = U(y_k) + S'_k + S''_k \leqslant 0.03n^2 + 0.36n^2 + 0.1n^2 < 0.5n^2,$$

and (4.3) holds also in Case 2.

Case 3. $n/3 < k \leqslant n/2$.

(a) When $1 \leqslant j \leqslant 5n/6$, then $2\pi/3 \geqslant (k+j)\pi/2n \geqslant \pi/6$, $\sin((k+j)\pi/2n) \geqslant 1/2$, and $|k-j|\,\pi/2n \leqslant \pi/4$.

Thus

$$|y_k - y_j| \geqslant 2\frac{4}{\pi}\frac{\sqrt{2}}{2}\frac{|k-j|}{n}\frac{\pi}{2}\frac{1}{2} = \sqrt{2}\,\frac{|k-j|}{n},$$

and

$$\begin{aligned} S'_k = \sum_{\substack{j=1\\ j\neq k}}^{5n/6} \frac{1}{|y_k - y_j|} &\leqslant 0.71\,\frac{n}{|k-j|} \\ &\leqslant 0.71n((1+1/2+\cdots+1/k)+(1+1/2+\cdots+1/(n-k))) \\ &\leqslant 0.71n(2+2\log(n/2)) \leqslant 0.38n^2, \end{aligned}$$

where we used the inequality

$$\begin{aligned} (1/2+\cdots+1/k)+(1/2+\cdots+1/(n-k)) &\leqslant \log k + \log(n-k) \\ &= \log(k(n-k)) \\ &\leqslant 2\log(n/2), \qquad 2 \leqslant k \leqslant n-2. \end{aligned}$$

(b) When $j \geqslant 5n/6$, $1/|y_k - y_j| \leqslant 1.2$ and

$$S''_k = \sum_{j \geqslant 5n/6+1}^{n-1} \frac{1}{|y_k - y_j|} \leqslant \frac{n}{6}\,1.2 = 0.2n \leqslant 0.02n^2.$$

Hence

$$|p'(y_k)| \leqslant A_1(y_k) \leqslant U(y_k) + S'_k + S''_k \leqslant 0.03n^2 + 0.38n^2 + 0.02n^2 \leqslant 0.43n^2,$$

and (4.3) holds also in Case 3. This completes the proof of Theorem 3.

Finally we prove Theorem 1 in the case $m=1$. We already know that (3.13) holds when $m=1$, and since $0 \in I_{n,1}$, according to Lemma 3(d), (3.14) is obtained from

$$\sup_{0 \leqslant x \leqslant 1} |\varphi'(x) - p'_{n-1}(x)| \leqslant \frac{\|\varphi^{(n)}\|_n}{n!} \sup_{0 \leqslant x \leqslant 1} \frac{|L'_n(x)|}{2^{n-1}n} = \frac{\|\varphi^{(n)}\|_n}{n!}\,\frac{T'_n(1)}{2^{n-1}n}, \tag{4.14}$$

where the last equality follows from (2.37) and (2.35). Now Theorem 1 can be proved from (3.14) and (3.15) in the same way as before, also with $m=1$.

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